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Stochastic Processes and their Applications 122 (2012) 3619–3647

stochastic processes and their applications

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On absolutely continuous compensators and nonlinear filtering equations in default risk models☆

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Received 27 April 2011; received in revised form 3 May 2012; accepted 2 July 2012
Available online 10 July 2012

Abstract

We discuss the pricing of defaultable assets in an incomplete information model where the default time is given by a first hitting time of an unobservable process. We show that in a fairly general Markov setting, the indicator function of the default has an absolutely continuous compensator. Given this compensator we then discuss the optional projection of a class of semimartingales onto the filtration generated by the observation process and the default indicator process. Available formulas for the pricing of defaultable assets are analyzed in this setting and some alternative formulas are suggested.

Keywords: Azéma supermartingale; Default indicator; Absolutely continuous compensators; Pricing of default risk; Nonlinear filtering; Zakai equation; Kushner–Stratonovich equation

1. Introduction

The motivation of this paper comes from a special field of Finance Theory; namely, the valuation of credit derivatives. The key problem in this field is to determine the price of an asset subject to default. To make the discussion more concrete, let us consider the basic financial instrument with default risk, which is a corporate bond with maturity T that pays the owner F units of a currency if the firm does not default until time T. If firm defaults before time T, usually there is a nonzero rebate, R, paid to the bond holder. Given this basic structure, the price of the defaultable bond at time t is given by the conditional expectation

☆ I would like to thank the Associate Editor and two anonymous referees for their comments which improved both the content and the presentation of the paper.

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0304-4149/$ - see front matter
doi:10.1016/j.spa.2012.07.001

$$\mathbb{E}[R\mathbf{1}_{[\tau\le T]} + F\mathbf{1}_{[\tau>T]}|\mathcal{G}_t],$$

where $\mathcal{G}$ is the *market's filtration* and the expectation is taken with respect to the martingale measure chosen by the market. Default time, τ, associated to the firm issuing the defaultable bond is often modeled as the first hitting time of barrier by a stochastic process representing the firm value. Leland [28] shows under certain conditions that it is optimal for the equity owners to liquidate the firm, and thus declare default, when the firm value falls below a barrier. On the other hand, the market is not able to identify the firm value continuously in time but has only a noisy observation of it. However, it is reasonable to assume that whether the default has occurred is directly observed in the market. In the simplest setting the process Y that satisfies

$$Y_t = B_t + \int_0^t b(X_s)\,ds,$$

can be viewed as the noisy observation of the firm value with B being the noise, independent of the firm value, and X is the firm's value process. Various aspects of this incomplete information issue have been studied in the literature. We can mention Jarrow and Turnbull [23], Lando [27], Duffie and Singleton [13], Kusuoka [26], Duffie and Lando [11], Nakagawa [31], Bielecki and Rutkowski [3], Çetin, et al. [6], Jarrow and Protter [21], Jarrow, et al. [22], Coculescu, et al. [7], and Campi and Çetin [4] to name a few. Frey and Runggaldier [15], Frey and Schmidt [16] and Frey and Schmidt [17] model credit risk from a nonlinear filtering point of view.

Valuation formulas for defaultable assets are given in different contexts in the literature. Duffie et al. [12] have given a formula that computes the price in the form of a stochastic discounting. In case of zero-coupon defaultable bond, i.e. $R = 0$ and $F = 1$ in above formulation, the time t price of this bond on the event $[\tau > t]$ is given by

$$J_t - \mathbb{E}\left[\mathbf{1}_{[t<\tau\le T]}\Delta J_\tau|\mathcal{G}_t\right], \quad t \in [0, T], \tag{1.1}$$

where

$$J_t = \mathbb{E}\left[\exp\left(-\int_t^{T\wedge\tau} \lambda_s\,ds\right)\middle|\mathcal{G}_t\right],$$

and λ is the so called *default intensity* which appears in the $\mathcal{G}$-canonical decomposition of the supermartingale $(\mathbf{1}_{[\tau>t]})_{t\ge0}$. More precisely, $(\lambda_{t\wedge\tau})_{t\ge0}$ is what makes

$$\left(\mathbf{1}_{[\tau>t]} + \int_0^{\tau\wedge t} \lambda_s\,ds\right)_{t\ge0} \tag{1.2}$$

a $\mathcal{G}$-martingale and $\left(\int_0^{\tau\wedge t} \lambda_s\,ds\right)_{t\ge0}$ is said to be the *compensator* of $(\mathbf{1}_{[\tau>t]})_{t\ge0}$. It is important to note here that such λ may not exist for any given random time τ. Although the formula in (1.1) is appealing in the sense that the price is a discounted expected value where the discounting factor is given by the default intensity, its drawback lies in the difficulty of computing the second term in (1.1) even if one is content with the assumption for the existence of an absolutely continuous compensator. In general it is not possible to compute the conditional expectation of the jump term appearing in the formula (see Çetin, et al. [6] for a special case when this computation is feasible). This led various authors suggest different formulas for the pricing of defaultable bonds.

An alternative formula to (1.1) for the price of a zero-coupon defaultable bond before default is given by

$$Z_t^{-1}\mathbb{E}[Z_T|\mathcal{F}_t^Y], \tag{1.3}$$

where Z is the so-called *Azéma supermartingale* defined by $Z_t := \mathbb{P}[\tau > t | \mathcal{F}_t^Y]$. One should mention at this point the works of Collin-Dufresne, et al. [10], Bielecki, et al. [2], Coculescu, et al. [8] and Coculescu and Nikeghbali [9] as good references that are attempting to solve the valuation problem in the general case. The papers [8,9,2] also contain a discussion of several approaches to obtain the valuation formula.

The main assumption in the formulas which compute the price as a discounted conditional expectation in the works listed above, and in many others, is that the increasing process $(\mathbf{1}_{[\tau \leq t]})_{t \geq 0}$ has an absolutely continuous compensator leading to the canonical decomposition described in (1.2). This assumption has found widespread use in models of credit risk due to intuitive representation of λ as the probability of default in the next instant (see [11] for the relation between λ and credit spreads). In a recent paper, Janson et al. [20] have identified a set of natural sufficient conditions under which $(\mathbf{1}_{[S \leq t]})_{t \geq 0}$ has an absolutely continuous compensator for *any* totally inaccessible stopping time S with respect to the natural filtration of a Markov process from a certain class.

The present paper has two main objectives. In Section 2, we show that, under natural regularity conditions, $(\mathbf{1}_{[\tau \leq t]})_{t \geq 0}$ has an absolutely continuous $\mathcal{G}$-compensator when τ is the first hitting time of 0 for the diffusion

$$X_t = X_0 + W_t + \int_0^t a(X_s)\, ds \tag{1.4}$$

and the observation process is given by

$$Y_t = B_t + \int_0^t b(s, X_s)\, ds, \tag{1.5}$$

where B and W are independent standard Brownian motions. More precisely, we show the existence of an $\mathcal{F}^Y$-adapted process $(\lambda_t)_{t \geq 0}$ such that the process in (1.2) is a $\mathcal{G}$-martingale, where $\mathcal{G}$ is, as usual, the smallest filtration satisfying usual conditions that contains $\mathcal{F}^Y$ and make τ a stopping time. Modeling the default time as the first hitting time of a stochastic process is desirable since it is consistent with economic intuition that the equity owners are likely to declare default when the firm value falls below a certain level as we mentioned before. However, the disadvantage of this choice when the underlying stochastic process is continuous is that the default time becomes a predictable stopping time in the natural filtration of the underlying so that it does not admit an intensity. We refer the reader to the discussion in [21] for the problems with the default time being predictable. Our results show that although the first hitting time of a continuous diffusion is a predictable stopping time, if we shrink the filtration under the more reasonable assumption that the firm value can only be observed with some noise, the default time becomes a totally inaccessible stopping time and admit an intensity. We will see that the finite variation part of the Doob–Meyer decomposition of Z is absolutely continuous, which will in turn imply the existence of λ leading to (1.2). An explicit representation for λ is also given. We achieve this by computing the canonical representation of the associated Azéma supermartingale using tools from non-linear filtering. We remark here that the results of Janson et al. [20] are not applicable to yield an absolutely continuous compensator since τ is not a totally inaccessible stopping time, in fact it is predictable, in the natural filtration of X and it is, in general, not a stopping time with respect to the filtration generated by Y. Thus, our results indicate that the existence of a default intensity requires much weaker conditions when there is only a noisy information on the fundamental processes that drive the default event. As for the pricing of defaultable securities, the existence of an absolutely continuous compensator implies that one can use the formulas in the aforementioned works which assume its existence. Moreover, at the end of Section 2, we will suggest some alternatives to the formula given in (1.1).

In view of the results in Section 2, we solve in Section 3 the nonlinear filtering problem corresponding to the $\mathcal{G}$-optional projection of semimartingales. In particular, we obtain the *Kushner–Stratonovich equations* for the $\mathcal{G}$-conditional distribution of X. As a by product, this suggests yet another alternative formula to price defaultable bonds. Another use of the solution to this filtering problem is that it immediately gives us the explicit semimartingale decomposition of the price processes of defaultable assets, which are in general not easy to compute using, e.g., the formula (1.1) mentioned above. On the way to the solution of the filtering problem, we also briefly discuss a common assumption in default risk models, the so-called (H)-hypothesis, due to its connection to a certain *martingale representation result* which was essential in our proof of equations of nonlinear filtering. As an application of the filtering equations, the Doob–Meyer decomposition for the value process of the rebate is calculated and the equation of extrapolation is given. An extension of the filtering equations to a non-Markovian setting is also discussed at the end of Section 3.

Finally, it is worth emphasizing that the setup considered in Sections 2 and 3 and the specific filtering problem studied in Section 3 cannot be viewed within the standard class of filtering problems with jump–diffusion observation which have been previously studied and applied to credit risk (see [15,5]). In these models the default times are the jump times of a marked point process or a jump–diffusion and as such they are *totally inaccessible* stopping times, with respect to the large filtration to which all the processes are adapted, and admit an intensity. As a consequence, in every shrinkage of the filtration, the default indicator processes will continue to have absolutely continuous compensators. However, in our setup the default time, being the first hitting time of a continuous diffusion, is a *predictable* stopping time in the large filtration and, thus, does not admit an intensity. Moreover, it is not a priori clear how much one needs to shrink the large filtration in order to make the default time a totally inaccessible stopping time. These considerations make it impossible to represent our filtering problem within the framework of the above models. Consequently, one needs to develop a different approach and in Sections 2 and 3 we follow the one that is outlined above.

2. Existence of an absolutely continuous compensator

Let B and W be two independent standard Brownian motions with $B_0 = W_0 = 0$ defined on $(\Omega, \mathcal{F}, (\mathcal{H}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses. All processes in this and subsequent sections will be defined on this filtered probability space. Observe that $\mathcal{H}$ is allowed to be strictly larger than the filtration generated by B and W.

Suppose X is a diffusion which is a strong solution to

$$X_t = X_0 + W_t + \int_0^t a(X_s)\,ds, \tag{2.1}$$

where $X_0 > 0$ is an $\mathcal{H}_0$-measurable random variable with $\mathbb{P}(X_0 \in dx) = \mu(dx)$ where μ is a probability measure on the Borel subsets of $(0, \infty)$.

Assumption 1. $\mathbb{E}X_0^2 < \infty$ and the function $a : \mathbb{R} \mapsto \mathbb{R}$ satisfies the following:

1. a is continuously differentiable with a bounded derivative.

2. $\lim_{x\to\infty} A(x)$ exists, possibly infinite, where
$$A(x) := \int_0^x a(y)\,dy.$$
3. $a(\infty) := \lim_{x\to\infty} a(x)$ exists (possibly infinite). If $a(\infty) = -\infty$ then there exist some $K_a > 0$ and $g_a \geq 0$ such that for any $x \geq g_a$
$$a(x) = -K_a x + f_a(x)$$
where f_a is a negative function such that $-\int_0^x f_a(y)\,dy \leq c_f x^p$ for some $p < 2$.

Remark 1. Under Assumption 1 there exists a unique strong solution to (2.1) such that for every $T > 0$, $\mathbb{E}X_t^2 \leq \gamma(1 + \mathbb{E}X_0^2)e^{\gamma t}$ for all $t \in [0, T]$ for some constant γ that depends only on T and the upper bound on the derivative of a (see Theorem 5.2.9 in [25]).

Remark 2. The assumption on the asymptotic behavior of a is to ensure that a does not make unbounded oscillations when it diverges to $-\infty$. This will be used in obtaining bounds on the density of the first hitting time of 0 by X below. Note that this assumption is satisfied when X is a Gaussian process, i.e. when a is affine.

Let
$$\tau := \inf\{t > 0 : X_t = 0\}$$
and define
$$H^a(t, x) := P_x[\tau > t], \tag{2.2}$$
where P_x is the law of the solutions of (2.1) with $X_0 = x$. Observe that τ is a predictable $\mathcal{H}$-stopping time. As such, the $\mathcal{H}$-compensator of the process $(\mathbf{1}_{[\tau>t]})_{t\geq 0}$ is the process itself. Our main goal in this section is to show that under a particular shrinkage of the filtration, this process will have an absolutely continuous compensator.

We will show in the theorem below that
$$H^a(t, x) = 1 - \int_0^t \ell^a(u, x)\,du,$$
for some function ℓ^a along with some further properties of the density which will be useful in the sequel for the existence of an absolutely continuous compensator. Recall that when $a \equiv 0$
$$\ell^a(t, x) = \ell(t, x) := \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right),$$
for $x > 0$, which is the probability density function of the first hitting time of 0 for a standard Brownian motion started at x. We will also drop the superscript in H^a when $a \equiv 0$ for notational convenience.

Theorem 2.1. *Let H^a be as in (2.2). Then, under Assumption 1,*

1. *H^a is absolutely continuous. That is, there exists a function ℓ^a such that*
$$H^a(t, x) = 1 - \int_0^t \ell^a(u, x)\,du, \quad \forall x > 0.$$
Moreover, $H^a(t, x) > 0$ for all $t \geq 0$ and $x > 0$.

2. *Let*

$$\delta := \sup_{x\in\mathbb{R}_+} \frac{x}{e^{\frac{x}{6}} - e^{-\frac{5x}{6}}}$$

and K_g *be the smallest constant,* K*, for which* $|a(x)| \leq K(1+|x|)$ *for all* $x \in \mathbb{R}$*. Then,*

$$\int_0^\infty \frac{1}{s}\ell^a(s,x)\,ds \leq 2\delta^{3/2}\frac{1+K_g x}{x^2}. \tag{2.3}$$

3. *The mapping* $t \mapsto t\ell^a(t,x)$ *is locally bounded uniformly in* x*.*

Proof. See Appendix. □

In addition to X there is also an *observation process* Y which is defined by

$$Y_t = B_t + \int_0^t b(s, X_s)\,ds \tag{2.4}$$

and $b : \mathbb{R}_+ \times \mathbb{R}^2 \mapsto \mathbb{R}$ is satisfying the following:

Assumption 2. $b(t,0) = 0$ for all $t \geq 0$. Moreover, b is locally Lipschitz in x, thus, for every $T > 0$ there exists a $K_b(T)$ such that $|b(t,x)| \leq K_b(T)|x|$ for all $t \leq T$.

Note that the assumption $b(t,0) = 0$ for all t is without loss of generality since the filtrations generated by Y or $Y - \int_0^{\cdot} b(s,0)\,ds$ are the same.

In this section, we are mainly interested in the *Azéma supermartingale*

$$Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t^Y], \tag{2.5}$$

where $\mathcal{F}^Y$ is the minimal filtration satisfying the usual conditions generated by Y. As the conditional expectation is only defined almost surely for each t, Z is defined to be the unique $\mathcal{F}^Y$*-optional projection* of $(\mathbf{1}_{[\tau>t]})_{t\geq 0}$. We recall the definition of optional projection here for the convenience of the reader.

Definition 2.1. Let U be a positive or bounded measurable process and $(\mathcal{F}_t)$ be a filtration satisfying the usual conditions. Then, the $(\mathcal{F}_t)$-optional projection of U is the $(\mathcal{F}_t)$-optional process V such that for any $\mathcal{F}$-stopping time S the following holds:

$$\mathbb{E}[U_S\mathbf{1}_{[S<\infty]}|\mathcal{F}_S] = V_S\mathbf{1}_{[S<\infty]}.$$

The above definition, taken from Section 5 in Chapter IV of [33], has an obvious extension to integrable measurable processes. We emphasize here that this choice of optional projection will be made without notice whenever we consider processes defined by projection onto a smaller filtration, in particular when we consider the filtering of a signal by an observation process.

Z, being a $(\mathbb{P}, \mathcal{F}^Y)$-supermartingale, has a càdlàg modification due to the continuity of the map $t \mapsto \mathbb{P}[\tau > t]$, (see Theorem 2.9 in Chapter II of [33]), which we will use henceforth. Note that Z is a nonnegative supermartingale of class D. The Doob–Meyer decomposition for such supermartingales (see Theorem 8 in Chapter III of [32]) gives the following.

Proposition 2.1. *There exists a unique increasing and* $\mathcal{F}^Y$*-predictable process* C *with* $C_0 = 0$ *such that* $Z + C$ *is a uniformly integrable* $(\mathbb{P}, \mathcal{F}^Y)$*-martingale.*

In the rest of this section, we will compute the above decomposition explicitly and discuss some of its consequences. The following is the main result of this section whose lengthy proof is delegated to Appendix.

Theorem 2.2. *Let Z be the Azéma supermartingale given by (2.5), and C be as in Proposition 2.1. Then, under Assumptions 1 and 2 the following holds:*

1. *Z is a.s. strictly positive and for any $t \geq 0$*

$$Z_t = \mathbb{E}[H^a(t, X_0)] + \int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]} H^a(t-s, X_s)\left(b(s, X_s) - \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\right)|\mathcal{F}_s^Y\right] dB_s^Y, \quad (2.6)$$

where

$$B_t^Y = Y_t - \int_0^t \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\, ds$$

is an $\mathcal{F}^Y$-Brownian motion.

2. *$C_t = \int_0^t c_s\, ds$, where*

$$c_t = \int_0^\infty \ell^a(t, x)\mu(dx) + \int_0^t \mathbb{E}\Big[\mathbf{1}_{[\tau>s]} \ell^a(t-s, X_s) \times \left(b(s, X_s) - \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\right)|\mathcal{F}_s^Y\Big] dB_s^Y,$$

and μ corresponds to the initial distribution of X_0.

3. *$Z + C$ is a uniformly integrable $(\mathbb{P}, \mathcal{F}^Y)$-martingale defined by $Z_t + C_t = \int_0^t \eta_s\, dB_s^Y$, where*

$$\eta_t := \mathbb{E}[\mathbf{1}_{[\tau>t]} b(t, X_t)|\mathcal{F}_t^Y] - Z_t \mathbb{E}[b(t, X_t)|\mathcal{F}_t^Y].$$

Proof. See Appendix. □

The next remark is considering a possible relaxation of the independence assumption on B and W. However, as it heavily relies on a certain argument in the proof of Theorem 2.2, the reader is invited to read the following remark along with the proof of the preceding theorem.

Remark 3. A natural question at this point is 'how important is the independence assumption on B and W?' To this end let us suppose $X = X_0 + W$ and $d[B, W]_t = \varrho_t\, dt$ where ϱ is a progressively measurable process and X_0 is a strictly positive constant. Repeating what we did in the proof of Theorem 2.2 yields

$$Z_t = H^a(t, X_0) + \int_0^t \mathbb{E}[\mathbf{1}_{[\tau>s]} H_x(t-s, X_s)\varrho_s|\mathcal{F}_s^Y]\, dB_s^Y + \int_0^t \left\{\mathbb{E}\left[\mathbf{1}_{[\tau>s]} H(t-s, X_s)\left(b(s, X_s) - \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\right)|\mathcal{F}_s^Y\right]\right\} dB_s^Y.$$

Thus, in order to arrive at a similar decomposition we obtained in Parts 2 and 3 of Theorem 2.2, we will need some assumptions on the correlation coefficient ϱ. Indeed, if B and W are the same Brownian motions, i.e. $\varrho \equiv 1$, it is clear that $Z = \mathbf{1}_{[\tau>t]}$ and its compensator is itself. On the other hand, if $|\varrho_t| \leq \varrho |X|_t^2$ for some constant $\varrho \geq 0$ (at least when X is within some open interval including 0), then

$$\mathbf{1}_{[\tau>s]}|\varrho_s \ell_x(t-s, X_s)| \leq \mathbf{1}_{[\tau>s]} \varrho\, \ell(t-s, X_s)\left(X_s + \frac{X_s^3}{t-s}\right).$$

Thus, using the explicit form of ℓ, one can repeat the arguments that led to the explicit Doob–Meyer decomposition in Theorem 2.2 to justify the interchange of ordinary and stochastic integrals and establish that C is absolutely continuous. Observe that by placing this assumption on ϱ, what we in fact require is that the correlation coefficient between two Brownian motions is vanishing quite fast when X is approaching to 0, i.e. B and W are behaving almost independently when X is in a neighborhood of 0. It would be interesting to investigate whether such a condition is a necessary condition for C to be absolutely continuous.

In Section 1, we claimed that the absolute continuity of C would lead to $(\mathbf{1}_{[\tau>t]})_{t\geq 0}$ having an absolutely continuous compensator. We are now in a position to make this precise and show that it is indeed the case. To this end, let $\mathcal{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by D and Y, and augmented with the $\mathbb{P}$-null sets, where $D_t := \mathbf{1}_{[\tau>t]}$. Then, D is a $\mathcal{G}$-adapted càdlàg $\mathbb{P}$-supermartingale and there exists a $\mathcal{G}$-predictable Λ with $\Lambda_0 = 0$ such that $D + \Lambda$ is a $(\mathbb{P}, \mathcal{G})$-martingale. The $\mathcal{F}^Y$-decomposition of Z allows us to compute Λ directly as follows:

Corollary 2.1. *Under the assumptions of Theorem 2.2, $D + \Lambda$ is a $(\mathbb{P}, \mathcal{G})$-martingale such that $d\Lambda_t = \mathbf{1}_{[\tau\geq t]}\lambda_t dt$ where*

$$\lambda_t = \frac{\int_0^\infty \ell^a(t,x)\mu(dx) + \int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}\ell^a(t-s,X_s)\left(b(s,X_s) - \mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\Big|\mathcal{F}_s^Y\right]dB_s^Y}{\int_0^\infty H^a(t,x)\mu(dx) + \int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}H^a(t-s,X_s)\left(b(s,X_s) - \mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\Big|\mathcal{F}_s^Y\right]dB_s^Y},$$

and μ is the probability distribution of X_0.

Proof. It is well-known (see, e.g., Theorem 3.4 in [8]) that

$$\lambda_t = \frac{1}{Z_{t-}}\frac{dC_t}{dt}.$$

The result now follows from Theorem 2.2, and that

$$\mathbb{E}[H^a(t,X_0)] = \int_0^\infty H^a(t,x)\mu(dx). \quad \square$$

Given the above formulation of Z we have the following representation formula as a consequence of, e.g., Proposition 2.3 in Chapter IX of [33].

Corollary 2.2. *Under the assumptions of Theorem 2.2,*

$$Z_t = \exp\left(-\int_0^t \lambda_s\, ds\right)\xi_t^{-1}\kappa_t, \quad \text{where}$$
$$\xi_t = \exp\left(\int_0^t \mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\, dY_s - \frac{1}{2}\int_0^t \mathbb{E}^2[b(s,X_s)|\mathcal{F}_s^Y]\, ds\right),$$
$$\kappa_t = \exp\left(\int_0^t \frac{\mathbb{E}[\mathbf{1}_{[\tau>s]}b(s,X_s)|F_s^Y]}{Z_s}dY_s - \frac{1}{2}\int_0^t \frac{\mathbb{E}^2[\mathbf{1}_{[\tau>s]}b(s,X_s)|F_s^Y]}{Z_s^2}\, ds\right).$$

Proof. Note that

$$dZ_t = \mathbb{E}[\mathbf{1}_{[\tau>t]}b(t,X_t)|\mathcal{F}_t^Y]dB_t^Y - Z_t(\lambda_t dt + \mathbb{E}[b(t,X_t)|\mathcal{F}_t^Y]dB_t^Y).$$

Thus, it follows from Proposition 2.3 in Chapter IX of [33] that

$$Z_t = \xi_t^{-1}\exp\left(-\int_0^t \lambda_s\, ds\right)\left(1 + \int_0^t \xi_s \exp\left(\int_0^s \lambda_r\, dr\right)\mathbb{E}[\mathbf{1}_{[\tau>s]}b(s,X_s)|F_s^Y]\, dY_s\right),$$

where

$$\xi_t = \exp\left(\int_0^t \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\, dY_s - \frac{1}{2}\int_0^t \mathbb{E}^2[b(s, X_s)|\mathcal{F}_s^Y]\, ds\right).$$

Let

$$\kappa_t := \left(1 + \int_0^t \xi_s \exp\left(\int_0^s \lambda_r\, dr\right) \mathbb{E}[\mathbf{1}_{[\tau>s]} b(s, X_s)|F_s^Y]\, dY_s\right)$$

and observe that κ is strictly positive with $d\kappa_t = \frac{\kappa_t}{Z_t}\mathbb{E}[\mathbf{1}_{[\tau>t]} b(t, X_t)|F_t^Y]\, dY_t$, i.e.

$$\kappa_t = \exp\left(\int_0^t \frac{\mathbb{E}[\mathbf{1}_{[\tau>s]} b(s, X_s)|F_s^Y]}{Z_s} dY_s - \frac{1}{2}\int_0^t \frac{\mathbb{E}^2[\mathbf{1}_{[\tau>s]} b(s, X_s)|F_s^Y]}{Z_s^2}\, ds\right). \quad \square$$

Remark 4. The above corollary also gives the multiplicative decomposition of Z as a product of a local martingale and a decreasing process. Indeed, it is a straightforward application of integration by parts formula to see that $\xi^{-1}\kappa$ is a $(\mathbb{P}, \mathcal{F}^Y)$-local martingale. Observe that one can obtain the multiplicative decomposition directly from the Doob–Meyer decomposition of Z since $Z = ne^{-\int_0^\cdot \lambda_s\, ds}$ where $dn = e^{\int_0^\cdot \lambda_s\, ds}(dZ + dC)$ with $n_0 = 1$, and C is as defined in Proposition 2.1.

We now will take a detailed look at the formula in (1.1). Recall that the expression in (1.1) equals

$$S_t := \mathbb{P}[\tau > T|\mathcal{G}_t] \tag{2.7}$$

on the set $[\tau > t]$.

Our aim in the rest of this section is to obtain alternative representations for S which will emphasize the role of default intensity as a stochastic discount factor as observed in the Introduction. These representations will be obtained via equivalent changes of probability measure and our first change of measure will be defined by the process M given by

$$M_t := \exp\left(\int_0^t b(s, X_s) dY_s - \frac{1}{2}\int_0^T b^2(s, X_s)\, ds\right). \tag{2.8}$$

Observe that

$$dM_t^{-1} = M_t^{-1} b(t, X_t)\, dB_t,$$

and, thus, M^{-1} is a strictly positive $(\mathbb{P}, \mathcal{H})$-martingale due to the fact that (X, Y) is a non-explosive solution to (2.1) and (2.4); see, e.g., Exercise 2.10 in Chapter IX of [33]. Therefore, for each $t > 0$ one can define a probability measure $\mathbb{Q}_t$ on $\mathcal{H}_t$ such that

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = M_t^{-1},$$

where $\mathbb{P}_t$ is the restriction of $\mathbb{P}$ to $\mathcal{H}_t$. Under $\mathbb{Q}_t$, $(Y_s)_{s\in[0,t]}$ is a standard Brownian motion independent of $(X_s)_{s\in[0,t]}$. The reason for defining a family of probability measures rather than a single $\mathbb{Q}$ valid on $\mathcal{H}_\infty$ is due to the fact that M^{-1} is not necessarily a uniformly integrable martingale in this infinite horizon setting. Nevertheless, for notational convenience we will drop the subscript in $\mathbb{Q}_t$ and write $\mathbb{Q}$ in what follows when no confusion arises.

A useful observation, which we will often make use of in the sequel, is that for any integrable and $\mathcal{H}_t$-measurable random variable F one has

$$\mathbb{E}[F|\mathcal{F}_t^Y] = \frac{\mathbb{E}^{\mathbb{Q}}[FM_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]}. \tag{2.9}$$

In particular, taking $F = M_t^{-1}$ yields

$$\mathbb{E}[M_t^{-1}|\mathcal{F}_t^Y] = \frac{1}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]}, \tag{2.10}$$

and $F = \mathbf{1}_{[\tau>t]}$ gives

$$Z_t = \frac{\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{[\tau>t]}M_t|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]}. \tag{2.11}$$

The next lemma is folklore in Stochastic Filtering Theory and would have followed from Theorem 8.1 in [30] if M were a square integrable $(\mathbb{Q}, \mathcal{H})$-martingale. Since the standard texts on Filtering Theory do not appear to be giving the proof for the general case, we nevertheless provide its proof in Appendix.

Lemma 2.1. *Under Assumptions 1 and 2 we have*

$$\xi_t = \mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y] = 1 + \int_0^t \mathbb{E}^{\mathbb{Q}}[M_s|\mathcal{F}_s^Y]\mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\,dY_s,$$
$$\xi_t^{-1} = \mathbb{E}[M_t^{-1}|\mathcal{F}_t^Y] = 1 - \int_0^t \mathbb{E}[M_s^{-1}|\mathcal{F}_s^Y]\mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\,dB_s^Y.$$

In view of the above lemma we see that ξ^{-1} is a $(\mathbb{P}, \mathcal{F}^Y)$-martingale. This leads to the following.

Corollary 2.3. *Let S be as in (2.7). Under Assumptions 1* and *2*

$$S_t = \mathbf{1}_{[\tau>t]}\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_s\,ds\right)\exp\left(\int_t^T \vartheta_s dY_s - \frac{1}{2}\int_t^T \vartheta_s^2\,ds\right)\middle|\mathcal{F}_t^Y\right],$$

where $\vartheta_s = \frac{\mathbb{E}[\mathbf{1}_{[\tau>s]}b(s,X_s)|F_s^Y]}{Z_s}$.

Proof. It is well-known that (see, e.g., Lemma 3.1 in [14])

$$\mathbb{E}[\mathbf{1}_{[\tau>T]}|\mathcal{G}_t] = \mathbf{1}_{[\tau>t]}\frac{\mathbb{E}[\mathbf{1}_{[\tau>T]}|\mathcal{F}_t^Y]}{Z_t} = \mathbf{1}_{[\tau>t]}\frac{\mathbb{E}[Z_T|\mathcal{F}_t^Y]}{Z_t}.$$

Since

$$\mathbb{E}[Z_T|\mathcal{F}_t^Y] = \frac{\mathbb{E}^{\mathbb{Q}}[Z_TM_T|\mathcal{F}_t^Y]}{\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]},$$

the claim follows from the representation in Corollary 2.2 and Lemma 2.1. $\square$

Remark 5. Similarly, one can get the following formula for any $F \in L^1(\mathcal{F}_T^Y, \mathbb{P})$:

$$\begin{aligned}&\mathbf{1}_{[\tau>t]}\mathbb{E}[F\mathbf{1}_{[\tau>T]}|\mathcal{G}_t]\\&\quad=\mathbf{1}_{[\tau>t]}\mathbb{E}^{\mathbb{Q}}\left[F\exp\left(-\int_t^T\lambda_s\,ds\right)\exp\left(\int_t^T\vartheta_s dY_s-\frac{1}{2}\int_t^T\vartheta_s^2\,ds\right)\bigg|\mathcal{F}_t^Y\right].\end{aligned}$$

The above corollary can be viewed as an alternative to the formula in (1.1). One advantage is that it does not require a computation of a jump term, in addition to the conditional expectation being taken with respect to arguably a simpler filtration, $\mathcal{F}^Y$. The price to pay in return is that the computation is made under a different, but equivalent, probability measure and S is not equal to the conditional expectation of $\exp\left(-\int_t^T\lambda_s\,ds\right)$ but that of its multiplication by a strictly positive deflator, κ_T/κ_t. Observe that κ is a strictly positive $(\mathbb{Q},\mathcal{F}^Y)$-local martingale. In case κ is a true martingale, we can make another change of probability measure and obtain the following result, which is a version of Proposition 4.3 in [9].

Corollary 2.4. *Suppose that $(\kappa_t)_{t\in[0,T]}$ is a $(\mathbb{Q},\mathcal{F}^Y)$-martingale and define $\tilde{\mathbb{Q}}$ on $\mathcal{F}_T^Y$ by setting $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}=\kappa_T$ on $\mathcal{F}_T^Y$. Then, under Assumptions 1 and 2,*

$$S_t=\mathbf{1}_{[\tau>t]}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\exp\left(-\int_t^T\lambda_s\,ds\right)\bigg|\mathcal{F}_t^Y\right].$$

Note that $\mathbb{P}\sim\tilde{\mathbb{Q}}$, too. As observed by [9] the above formula is in the spirit of the pricing formula of [10], who have obtained a pricing formula as an expectation of $\exp\left(-\int_t^T\lambda_s\,ds\right)$ but with respect to a probability measure which is only absolutely continuous with respect to $\mathbb{P}$. We refer the reader to the example in [9] that illustrates the difficulty with computing that expectation.

Remark 6. A sufficient condition for κ being a $(\mathbb{Q},\mathcal{F}^Y)$-martingale is the boundedness of b. Indeed, under this assumption

$$\frac{\left|\mathbb{E}\left[\mathbf{1}_{[\tau>s]}b(s,X_s)|F_s^Y\right]\right|}{Z_s}\leq\frac{\mathbb{E}\left[\mathbf{1}_{[\tau>s]}|b(s,X_s)||F_s^Y\right]}{Z_s}\leq K\frac{\mathbb{E}\left[\mathbf{1}_{[\tau>s]}|F_s^Y\right]}{Z_s}=K,$$

where K is an upper bound on $|b|$.

Remark 7. The formulas given above, in particular the expression for λ, often contains conditional expectations of the form $\mathbb{E}[\mathbf{1}_{[\tau>t]}F(X_t)|\mathcal{F}_t^Y]$ where F is a smooth function vanishing at 0. In general, it is not possible to compute such expectations since it is not possible to solve for the conditional distribution of X analytically. However, there are certain numerical methods that can be used to calculate these values. If we let

$$\rho_t F:=\mathbb{E}^{\mathbb{Q}}[M_t F(X_{t\wedge\tau})|\mathcal{F}_t^Y],$$

then it follows that

$$\mathbb{E}[\mathbf{1}_{[\tau>t]}F(X_t)|\mathcal{F}_t^Y]=\frac{\rho_t F}{\rho_t\mathbf{1}},$$

where $\mathbf{1}$ is the constant function that takes the value 1, whenever F is a smooth function vanishing at 0. Since the infinitesimal generators of X and X^τ are the same, the standard arguments from nonlinear filtering yield the *Zakai equation*

$$\rho_t F = \rho_0 F + \int_0^t \rho_s \mathcal{A} F\, ds + \int_0^t \rho_s b F\, dY_s \tag{2.12}$$

where

$$\mathcal{A} = a(x)\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}.$$

Numerical solution of (2.12) is beyond the scope of this paper. However, the *splitting-up* and *particle* methods which have been studied extensively in the literature can be applied to this setting to solve (2.12) numerically. The reader can find a lengthy discussion of these methods in Chapter 8 and 9 of [1] and the references therein.

Observe that although one can find the price of a defaultable asset by the formulas suggested above, they do not give immediately the $\mathcal{G}$-semimartingale decomposition of the price process. This is going to be the subject of the next section where we discuss the canonical decomposition of $\mathcal{G}$-optional projections of a class of $\mathcal{H}$-semimartingales. In particular, we obtain the $\mathcal{G}$-canonical decomposition of S in (3.9).

3. Nonlinear filtering equations for partially observable processes

In this section, we will investigate the 'optimal filters' of $\mathcal{H}$-adapted càdlàg processes when the available information is generated by the processes Y and D. As opposed to the previous section, we will restrict our attention to a finite horizon T. Recall from the previous section that $D_t = \mathbf{1}_{[\tau>t]}$, and we now set $\mathcal{G} = (\mathcal{G}_t)_{t\in[0,T]}$ (resp. $\mathcal{F}^Y = (\mathcal{F}^Y_t)_{t\in[0,T]}$) to be the filtration generated by $(D_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ (resp. $(Y_t)_{t\in[0,T]}$ only), and augmented with the $\mathbb{P}$-null sets. Let $\mathbb{Q} \sim \mathbb{P}_T$ be a probability measure on $(\Omega, \mathcal{H}_T)$ defined by the Radon–Nikodym density $\frac{d\mathbb{P}_T}{d\mathbb{Q}} = M_T$, where $\mathbb{P}_T$ is the restriction of $\mathbb{P}$ to $\mathcal{H}_T$ and M is as in (2.8). It follows that $(Y_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-Brownian motion independent of $(X_t)_{t\in[0,T]}$. This in particular implies that the natural filtration of Y is right continuous when augmented with $\mathbb{P}$-null sets and, thus, all $(\mathbb{P}, \mathcal{G})$ (resp. $(\mathbb{P}, \mathcal{F}^Y)$) martingales have right continuous versions, which we will use henceforth.

Note that in view of Theorem 2.2 and its Corollary 2.1 from Section 2

$$L_t := D_t - 1 + \int_0^{t\wedge\tau} \lambda_s\, ds \tag{3.1}$$

defines a $(\mathbb{P}, \mathcal{G})$-martingale with a single jump at τ of size -1. *In the rest of this section, we will assume that* Assumptions 1 *and* 2, *which yield in particular* Theorem 2.2 *and its* Corollary 2.1, *are in force without an explicit mention.*

We will obtain the filtering equations via an *innovations approach* (see Kallianpur [24] or Liptser and Shiryaev [30] for the background). To do this we need to obtain a martingale representation result for the square integrable $(\mathbb{P}, \mathcal{G})$-martingales. We will soon see that all such martingales can be written as a stochastic integral with respect to some $\mathcal{G}$-Brownian motion and L. The following is a well-known result in Filtering Theory.

Proposition 3.1. *Let*

$$\beta_t = Y_t - \int_0^t \mathbb{E}[b(s, X_s)|\mathcal{G}_s]\, ds, \quad t \in [0, T].$$

Then, β is a $(\mathcal{G}, \mathbb{P})$-Brownian motion.

We will next obtain a stochastic integral representation for the martingales in $\mathscr{M}$, where

$$\mathscr{M} = \{U : U \text{ is a square integrable } (\mathbb{P}, \mathcal{G})\text{-martingale}\}. \tag{3.2}$$

In credit risk models it is often assumed, in order to simplify the computations, that the following assumption, called (H)-Hypothesis, holds:

$$\text{Every } (\mathbb{P}, \mathcal{F}^Y)\text{-martingale is a } (\mathbb{P}, \mathcal{G})\text{-martingale.} \tag{H}$$

This assumption in particular implies a martingale representation property for square integrable $\mathcal{G}$-martingales; see [26]. The following is a well-known result taken from [8].

Theorem 3.1. *Every* $(\mathbb{P}, \mathcal{F}^Y)$*-martingale is a* $(\mathbb{P}, \mathcal{G})$*-martingale if and only if*

$$\mathbb{P}(\tau \le s | \mathcal{F}_t^Y) = \mathbb{P}(\tau \le s | \mathcal{F}_T^Y),$$

for every $s \le t \le T$.

It is not difficult to see that (H)-Hypothesis is not satisfied in general in our setting since Z has a non-zero martingale part in its canonical decomposition (see Theorem 3.3 in [8]). Nevertheless, we will have a predictable representation result for the martingales in $\mathcal{M}$ in the absence of this hypothesis in Proposition 3.3. Before the statement and a short proof of this result, we will prove a proposition which will show that the (H)-Hypothesis is satisfied (locally) under an equivalent probability measure as an aside. To this end, let us introduce the positive supermartingale

$$N_t = 1 - \int_0^t N_s \mathbb{E}[b(s, X_s) | \mathcal{G}_s] \, d\beta_s, \quad t \in [0, T].$$

Let $R_n := \inf \left\{ t > 0 : N_t > n \text{ or } N_t < \frac{1}{n} \right\}$ with the convention that $\inf \emptyset = T$. Note that since N is continuous, and strictly positive due to $\int_0^T \mathbb{E}[b^2(s, X_s)] \, ds < \infty$, $R_n \uparrow T$, $\mathbb{P}$-a.s. Associated to this stopping time, let $\mathcal{F}^n$ be the filtration generated by Y^{R_n}, augmented with the $\mathbb{P}$-null sets, and $\mathcal{G}^n$ be the smallest filtration containing $\mathcal{F}^n$ with respect to which τ is a stopping time.

Proposition 3.2. *Let* $\mathbb{P}^n$ *be the probability measure on* $(\Omega, \mathcal{G}_T)$ *defined by*

$$\frac{d\mathbb{P}^n}{d\mathbb{P}} = N_{R_n}.$$

Then, every $(\mathbb{P}^n, \mathcal{F}^n)$*-martingale is a* $(\mathbb{P}^n, \mathcal{G}^n)$*-martingale.*

Proof. Observe that $[Y^{R_n}, Y^{R_n}]_t = t \wedge R_n$ so that R_n is a $\mathcal{F}^n$-stopping time. Moreover, Y^{R_n} becomes a Brownian motion stopped at R^n under $\mathbb{P}^n$ while the canonical decomposition of D remains unchanged, i.e.

$$D = 1 + L - \Lambda,$$

where L, as defined by (3.1), is still a martingale under $\mathbb{P}^n$ and Λ is the continuous and increasing process defined in Corollary 2.1. Let Z^n denote the $\mathcal{F}^n$-optional projection of D under $\mathbb{P}^n$. Then, it follows from Theorem 8.1 in [30] that

$$Z_t^n = 1 - \int_0^t \mathbb{E}^n[\mathbf{1}_{[\tau > s]} \lambda_s | \mathcal{F}_s^n] \, ds,$$

where $\mathbb{E}^n$ is expectation with respect to $\mathbb{P}^n$ since Y^{R_n} has no drift, and $\langle L, Y\rangle \equiv 0$. Also observe that on $t < R_n$,

$$Z^n_t = 1 - \int_0^t \lambda_s \mathbb{E}^n[\mathbf{1}_{[\tau>s]}|\mathcal{F}^n_s]\,ds = 1 - \int_0^t \lambda_s Z^n_s\,ds,$$

since λ is adapted to $\mathcal{F}^Y$ in view of Corollary 2.1. As seen, Z^n is a continuous and decreasing process, and on $[t < R_n]$ it is given by

$$Z^n_t = \exp\left(-\int_0^t \lambda_s\,ds\right). \tag{3.3}$$

Moreover, it follows from Theorem 8.4 in [30] that, for $s \le t \le T$,

$$\mathbb{P}^n[\tau > s|\mathcal{F}_t] = Z^n_s.$$

The result now follows from Theorem 3.1. □

The next is the integral representation theorem that we are after.

Proposition 3.3. *Let $\mathscr{M}$ be as in (3.2). For any $U \in \mathscr{M}$ there exists a pair of $\mathcal{G}$-predictable process, $(\Phi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ such that*

$$U_t = U_0 + \int_0^t \Phi_s\,d\beta_s + \int_0^t \zeta_s dL_s.$$

Proof. It is clear that the vector semimartingale (Y, D) has the *weak predictable representation property* in the sense of Definition 13.13 in Chapter XIII of [19] for $(\mathbb{Q}, \mathcal{G})$-local martingales. Then it follows from Theorem 13.21 in Chapter XIII of [19] that it has the weak predictable representation property for $(\mathbb{P}, \mathcal{G})$-local martingales as well. This implies the claimed representation. □

We now return to solve the filtering problem when the observation is via the processes Y and D. Let us suppose that the unobserved signal, $P = (P_t)_{t\in[0,T]}$ is a $(\mathbb{P}, \mathcal{H})$-semimartingale such that

$$P_t = P_0 + \int_0^t V_s\,ds + m_t, \tag{3.4}$$

where m is a *continuous* $(\mathbb{P}, \mathcal{H})$-martingale and V is a measurable stochastic process adapted to $\mathcal{H}$ such that, $\mathbb{P}$-a.s.,

$$\int_0^t |V_s|\,ds < \infty,$$

for every $t \ge 0$. The solution of the filtering problem amounts to finding the semimartingale decomposition of the $(\mathbb{P}, \mathcal{G})$-optional projection of P, which will be denoted with $\hat{P}$. We make the following assumption on P.

Assumption 3. The semimartingale P in (3.4) satisfies the following:

1. $\sup_{t\le T} \mathbb{E}[P_t^2] < \infty$;
2. $\mathbb{E}\int_0^T V_s^2 ds < \infty$;
3. $m_t = \int_0^t \theta_s dB_s + n_t$ where θ is an $\mathcal{H}$-predictable process and n is a continuous $(\mathbb{P}, \mathcal{H})$-martingale strongly orthogonal to B.

In equations of nonlinear filtering (see, e.g. Theorem 8.1 in [30]), in order to obtain the filtering equations for the signal of the form $\int_0^\cdot V_s\,ds + m$, where m is a $(\mathbb{P},\mathcal{H})$-martingale, one needs the following useful fact, proof of which is the same as that of Lemma 8.4 in [30]; hence is omitted.

Proposition 3.4. *For any measurable and $\mathcal{H}$-adapted process V with the property*

$$\int_0^T \mathbb{E}[V_s^2]\,ds < \infty,$$

the random process

$$\left(\mathbb{E}\left[\int_0^t V_s\,ds\middle|\mathcal{G}_t\right] - \int_0^t \mathbb{E}\left[V_s|\mathcal{G}_s\right]\,ds\right)_{t\in[0,T]}$$

is a square integrable $(\mathbb{P},\mathcal{G})$-martingale.

The next theorem giving the semimartingale decomposition of $\hat{P}$ is the main result of this section. In what follows, we will write $\mathbb{E}[P_t|\mathcal{F}_t^Y,\tau=t]$ for the measurable function $\mathbb{E}[P_t|\mathcal{F}_t^Y,\tau]|_{\tau=t}$ (see Lemma A.1 in this respect).

Theorem 3.2. *Let P defined by (3.4) satisfy Assumption 3. Then*

$$\hat{P}_t = \hat{P}_0 + \int_0^t \hat{V}_s\,ds + \int_0^t \Phi_s\,d\beta_s + \int_0^t \zeta_s\,dL_s,$$

where

$$\begin{aligned}\Phi_t &= \hat{\theta}_t + \mathbb{E}[P_t b(t,X_t)|\mathcal{G}_t] - \hat{P}_t\mathbb{E}[b(t,X_t)|\mathcal{G}_t], \quad \textit{and}\\ \zeta_t &= \hat{P}_{t-} - \mathbb{E}[P_t|\mathcal{F}_t^Y,\tau=t],\end{aligned}$$

for every $t\in[0,T]$. In particular, the process $(\mathbb{E}[P_t|\mathcal{F}_t^Y,\tau=t])_{t\in[0,T]}$ is $\mathcal{G}$-predictable.

Proof. It follows from Proposition 3.4 that

$$\left(\mathbb{E}\left[\int_0^t V_s\,ds\middle|\mathcal{G}_t\right] - \int_0^t \mathbb{E}\left[V_s|\mathcal{G}_s\right]\,ds\right)_{t\in[0,T]}$$

is a square integrable $(\mathbb{P},\mathcal{G})$-martingale. Thus,

$$\left(\hat{P}_t - \int_0^t \hat{V}_s\,ds\right)_{t\in[0,T]}$$

is a square integrable $(\mathbb{P},\mathcal{G})$-martingale under Assumption 3, and, in view of Proposition 3.3, there exist $\mathcal{G}$-predictable processes, Φ and η, such that

$$\hat{P}_t = \hat{P}_0 + \int_0^t \hat{V}_s\,ds + \int_0^t \Phi_s\,d\beta_s + \int_0^t \zeta_s\,dL_s.$$

So, it remains to determine the processes Φ and ζ. Let $Y_t^n := Y_{t\wedge S_n}$ where $S_n := \inf\{t>0 : |Y_t| > n\}$. First note that, using integration by parts formula,

$$Y_t^n\hat{P}_t = \int_0^t \left\{Y_s^n\hat{V}_s + \mathbf{1}_{[s\le S_n]}(\Phi_s + \hat{P}_s\mathbb{E}[b(s,X_s)|\mathcal{G}_s])\right\}\,ds + n_t^1, \tag{3.5}$$

where n^1 is a $(\mathbb{P}, \mathcal{G})$-local martingale. We will next compute the optional projection of $Y^n P$ by directly taking the projection of

$$Y_t^n P_t = \int_0^t \left\{Y_s^n V_s + \mathbf{1}_{[s \le S_n]} P_s b(s, X_s)\right\} ds + \int_0^t Y_s^n \, dm_s \\ + \int_0^t \mathbf{1}_{[s \le S_n]} P_s \, dB_s + [m, B]_{t \wedge S_n}.$$

Thus,

$$Y_t^n \hat{P}_t = \int_0^t \left\{Y_s^n \hat{V}_s + \mathbf{1}_{[s \le S_n]} (\mathbb{E}[P_s b(s, X_s)|\mathcal{G}_s] + \hat{\theta}_s)\right\} ds + n_t^2, \tag{3.6}$$

where n^2 is a $(\mathbb{P}, \mathcal{G})$-local martingale. Equating (3.5) to (3.6) we get

$$\left(\int_0^{t \wedge S_n} \left\{\Phi_s + \hat{P}_s \mathbb{E}[b(s, X_s)|\mathcal{G}_s] - \hat{\theta}_s - \mathbb{E}[P_s b(s, X_s)|\mathcal{G}_s]\right\} ds\right)_{t \in [0,T]}$$

is a local martingale, thus, it must vanish since it is also predictable. We can in fact do similar calculations, after moving the origin from 0 to $r \in [0, T]$, to conclude that

$$\left(\int_r^{t \wedge S_n} \left\{\Phi_s + \hat{P}_s \mathbb{E}[b(s, X_s)|\mathcal{G}_s] - \hat{\theta}_s - \mathbb{E}[P_s b(s, X_s)|\mathcal{G}_s]\right\} ds\right)_{t \in [r,T]}$$

must vanish for every $r \ge 0$. Therefore, since $S_n \uparrow T$, $\mathbb{P}$-a.s.,

$$\Phi_t = \hat{\theta}_t + \mathbb{E}[P_t b(t, X_t)|\mathcal{G}_t] - \hat{P}_t \mathbb{E}[b(t, X_t)|\mathcal{G}_t], \quad t \in [0, T].$$

Now, we return to determine ζ. However, the filtering formula 4.10.8 in [29] yields

$$\zeta_t = \hat{P}_{t-} - \upsilon_t,$$

for some $\mathcal{G}$-predictable process υ, which is the unique $\mathcal{G}$-predictable process satisfying

$$\mathbb{E}\left[\int_0^T \nu_t P_t dD_t\right] = \mathbb{E}\left[\int_0^T \nu_t \upsilon_t dD_t\right], \tag{3.7}$$

for any bounded $\mathcal{G}$-predictable process ν. We will next show that $\upsilon = \left(\mathbb{E}[P_t|\mathcal{F}_t^Y, \tau = t]\right)_{t \in [0,T]}$. Before showing that the candidate process satisfies (3.7), let us first verify that it is $\mathcal{G}$-predictable.

In view of Lemma A.1, there exist appropriately measurable functions, f^1 and f^2 such that the $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-optional projections[1] of PM and M are given by $(f^1(\tau(\omega), \omega, t))_{t \in [0,T]}$ and $(f^2(\tau(\omega), \omega, t))_{t \in [0,T]}$, respectively. On the other hand, Bayes' formula yields for any $\mathcal{F}^{Y,\tau}$-stopping time S,

$$\mathbb{E}[P_S|\mathcal{F}_S^Y, \tau] = \frac{\mathbb{E}^{\mathbb{Q}}[P_S M_S|\mathcal{F}_S^Y, \tau]}{\mathbb{E}^{\mathbb{Q}}[M_S|\mathcal{F}_S^Y, \tau]} = \frac{f^1(\tau(\omega), \omega, S)}{f^2(\tau(\omega), \omega, S)};$$

i.e., $(\mathbb{P}, \mathcal{F}^{Y,\tau})$-optional projection of P is given by $(f(\tau(\omega), \omega, t))_{t \in [0,T]}$, where

$$f(u, \omega, t) := \frac{f^1(u, \omega, t)}{f^2(u, \omega, t)},$$

[1] $\mathcal{F}^{Y,\tau}$ is the smallest filtration satisfying the usual conditions and including $\mathcal{F}^Y$ such that $\sigma(\tau) \subset \mathcal{F}_0^{Y,\tau}$.

for $\omega \in \Omega$ and $u \geq 0$, $t \geq 0$. Note that $(f(\tau(\omega), \omega, t))_{t\in[0,T]}$ is $\mathcal{F}^{Y,\tau}$-optional since $(f^i(\tau(\omega), \omega, t))_{t\in[0,T]}$ is $\mathcal{F}^{Y,\tau}$-optional for $i = 1, 2$.

Moreover, since $f^i(t, \omega, t)$ is $\mathcal{F}^Y_t$-measurable for $i = 1, 2$ by Lemma A.1, we see that $f(t, \omega, t)$ is $\mathcal{F}^Y_t$-measurable for each $t \geq 0$, as well. Writing $\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]$ for $f(t, \omega, t)$, one has that $\left(\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]\right)_{t\in[0,T]}$ is a measurable and $\mathcal{F}^Y$-adapted process. By the definition of optional projections, the $(\mathbb{P}, \mathcal{F}^Y)$-optional projection of $\left(\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]\right)_{t\in[0,T]}$, denoted with u, satisfies $u_t = \mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]$ for every t. This implies that we can choose an $\mathcal{F}^Y$-optional version of $\left(\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]\right)_{t\in[0,T]}$. However, since Y is a Brownian motion after an equivalent change of measure, optional and predictable σ-algebras coincide yielding the $\mathcal{F}^Y$-predictability of $\left(\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t]\right)_{t\in[0,T]}$. Since $\mathcal{F}^Y$ is a sub-filtration of $\mathcal{G}$, the claim follows.

Now let us return to verify that u, the $\mathcal{F}^Y$-predictable (equivalently, $\mathcal{F}^Y$-optional) version of $(\mathbb{E}[P_t|\mathcal{F}^Y_t, \tau = t])_{t\in[0,T]}$, satisfies (3.7). Note that since $\mathcal{F}^Y$ is contained in $\mathcal{F}^{Y,\tau}$, u is $\mathcal{F}^{Y,\tau}$-optional as well. Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \nu_t P_t dD_t\right] &= -\mathbb{E}\left[\nu_\tau P_\tau \mathbf{1}_{[\tau\leq T]}\right] = -\mathbb{E}\left[\mathbf{1}_{[\tau\leq T]}\nu_\tau \mathbb{E}\left[P_\tau \middle| \mathcal{F}^{Y,\tau}_\tau\right]\right] \\
&= -\mathbb{E}\left[\mathbf{1}_{[\tau\leq T]}\nu_\tau f(\tau, \omega, \tau)\right] = \mathbb{E}\left[\int_0^T \nu_t f(t, \omega, t) dD_t\right] \\
&= \mathbb{E}\left[\int_0^T \nu_t u_t dD_t\right],
\end{aligned}$$

where the third equality follows from the definition of optional projections and the last equality holds since u is also $\mathcal{F}^{Y,\tau}$-optional and a version of $(f(t, \omega, t))_{t\in[0,t]}$. This concludes the proof. □

An immediate corollary to this theorem is the following result.

Corollary 3.1. *Let P defined by (3.4) satisfy Assumption 3 with $P_\tau = 0$. Then*

$$\begin{aligned}
\hat{P}_t = \hat{P}_0 &+ \int_0^t \hat{V}_s\, ds + \int_0^t \left\{\hat{\theta}_s + \mathbb{E}[P_s b(s, X_s)|\mathcal{G}_s] - \hat{P}_s \mathbb{E}[b(s, X_s)|\mathcal{G}_s]\right\} d\beta_s \\
&+ \int_0^t \hat{P}_{s-}\, dL_s.
\end{aligned}$$

Note that P vanishes at τ if $P = f(X)$ where f is a function that vanishes at 0. In view of this observation we will next establish a version of *Kushner–Stratonovich equation* (see Chapter 3 of [1] for the background) for the conditional distribution of X. To this end let $\mathbb{C}$ denote the class of continuous functions and $\mathbb{C}^2_{K,+}$ denote the class of twice continuously differentiable functions with a compact support in $(0, \infty)$ and define the operator $\mathcal{A} : \mathbb{C}^2_{K,+} \mapsto \mathbb{C}$ by

$$\mathcal{A}f(x) = a(x)f'(x) + \frac{1}{2}f''(x).$$

For any $f \in \mathbb{C}^2_{K,+}$ let

$$\pi_t f := \mathbb{E}[f(X_{t\wedge\tau})|\mathcal{G}_t].$$

Observe that π_t gives the $\mathcal{G}$-conditional distribution of X_t on the set $[\tau > t]$. Then, as an immediate corollary to Corollary 3.1, we have the following

Corollary 3.2. *Let* $f \in \mathbb{C}^2_{K,+}$. *Then,*

$$\pi_t f = \pi_0 f + \int_0^t \pi_s \mathcal{A} f \, ds + \int_0^t \{\pi_s f b - \pi_s f \pi_s b\} \, d\beta_s + \int_0^t \pi_{s-} f \, dL_s. \tag{3.8}$$

In particular, if P is the $(\mathbb{P}, \mathcal{H})$-martingale defined by $P_t = \mathbb{P}[\tau > T | \mathcal{H}_t] = \mathbf{1}_{[\tau > t]} H^a(T - t, X_t)$, where H^a is the function defined in (2.2), then $P_\tau = 0$, too. We also have that $\hat{P}_{t-} = D_{t-} \hat{P}_t$. Indeed, since

$$\mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{G}_s] = D_s \frac{\mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{F}^Y_s]}{Z_s}$$

we have that

$$\lim_{s \uparrow t} \mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{G}_s] = \frac{D_{t-}}{Z_t} \lim_{s \uparrow t} \mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{F}^Y_s].$$

However, $(\mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{F}^Y_s])_{s \in [0,T]}$ is a bounded $(\mathbb{P}, \mathcal{F}^Y)$-martingale, therefore it is continuous by Theorem 8.3.1 in [24] implying

$$\lim_{s \uparrow t} \mathbb{E}[D_s H^a(T - s, X_s) | \mathcal{G}_s] = \frac{D_{t-}}{Z_t} \mathbb{E}[D_t H^a(T - t, X_t) | \mathcal{F}^Y_t].$$

Hence, in view of the corollary above, one can write

$$\begin{aligned} \mathbb{P}[\tau > T | \mathcal{G}_t] &= \mathbb{E}[D_t H^a(T - t, X_t) | \mathcal{G}_t] \\ &= \mathbb{P}[\tau > T] + \int_0^t \mathbf{1}_{[s \le \tau]} \Big\{ \mathbb{E}[H^a(T - s, X_s) b(s, X_s) | \mathcal{G}_s] \\ &\quad - \mathbb{E}[H^a(T - s, X_s) | \mathcal{G}_s] \mathbb{E}[b(s, X_s) | \mathcal{G}_s] \Big\} d\beta_s \\ &\quad + \int_0^t \mathbf{1}_{[s \le \tau]} \mathbb{E}[H^a(T - s, X_s) | \mathcal{G}_s] dL_s. \end{aligned} \tag{3.9}$$

Note that the above formula also gives us the price of a defaultable zero-coupon bond which pays 1 unit of a currency to the holder at time-T in case default does not occur, and pays nothing if default does occur by time-T. As discussed in the introduction, there is usually a rebate paid to the bond holder in case of default. Let us suppose that the rebate is random and amounts to P_τ for some stochastic process P. Time-t value of the rebate is given by $\mathbb{E}[P_\tau \mathbf{1}_{[\tau \le T]} | \mathcal{G}_t]$. The next proposition gives us a decomposition for the value of the rebate before default happens.

Proposition 3.5. *Let* P *defined by* (3.4) *be bounded and satisfy Assumption 3. Then,* $(\mathbb{E}[P_\tau \mathbf{1}_{[t < \tau \le T]} | \mathcal{G}_t])_{t \in [0,T]}$ *has the unique Doob–Meyer decomposition*

$$\mathbb{E}[P_\tau \mathbf{1}_{[t < \tau \le T]} | \mathcal{G}_t] = \mathbb{E}[\alpha_T | \mathcal{G}_t] - \alpha_t,$$

where

$$\alpha_t = \int_0^{t \wedge \tau} \mathbb{E}[P_s | \mathcal{F}^Y_s, \tau = s] \lambda_s \, ds, \quad t \in [0, T].$$

Proof. Let

$$R_t := \mathbb{E}[P_\tau \mathbf{1}_{[t < \tau \le T]} | \mathcal{G}_t] = \mathbb{E}[P_\tau \mathbf{1}_{[\tau \le T]} | \mathcal{G}_t] - \mathbb{E}[P_\tau \mathbf{1}_{[\tau \le t]} | \mathcal{G}_t].$$

Then, $R_T = 0$ and

$$R_t = \left(\mathbb{E}[P_\tau^+ \mathbf{1}_{[\tau \le T]}|\mathcal{G}_t] - \mathbb{E}[P_\tau^+ \mathbf{1}_{[\tau \le t]}|\mathcal{G}_t]\right) - \left(\mathbb{E}[P_\tau^- \mathbf{1}_{[\tau \le T]}|\mathcal{G}_t] - \mathbb{E}[P_\tau^- \mathbf{1}_{[\tau \le t]}|\mathcal{G}_t]\right),$$

where x^+ (resp. x^-) denotes the positive (resp. negative) part of a real number x. The above implies R is the difference of two positive supermartingales, thus, by Theorem 8 in Chapter III of [32], there exists a predictable process, α, of finite variation with $\alpha_0 = 0$ such that $R - \alpha$ is a $(\mathbb{P}, \mathcal{G})$-martingale. Since $R_T = 0$, we thus have the unique decomposition of R as follows:

$$R_t = \mathbb{E}[\alpha_T|\mathcal{G}_t] - \alpha_t. \tag{3.10}$$

On the other hand, if we apply integration by parts formula to $D\hat{P}$ we obtain

$$d(D\hat{P})_t = D_{t-}\left\{\hat{V}_t - \mathbb{E}[P_t|\mathcal{F}_t^Y, \tau = t]\lambda_t\right\}dt + dn_t^1, \tag{3.11}$$

where n^1 is $(\mathbb{P}, \mathcal{G})$-local martingale. Moreover, since

$$d(DP)_t = D_t V_t dt + D_t dm_t - P_{t-} dD_t = D_{t-} V_t dt + D_{t-} dm_t - P_\tau \mathbf{1}_{[\tau \le t]},$$

by taking the optional projection of the above, we see that

$$D_t \hat{P}_t = \hat{P}_0 + \int_0^t D_{s-} \hat{V}_s\, ds - \mathbb{E}[P_\tau \mathbf{1}_{[\tau \le t]}|\mathcal{G}_t] + n_t^2, \tag{3.12}$$

where n^2 is a $(\mathbb{P}, \mathcal{G})$-local martingale. Therefore, comparing (3.11) to (3.12), we obtain that

$$\left(\mathbb{E}[P_\tau \mathbf{1}_{[\tau \le t]}|\mathcal{G}_t] - \int_0^{t\wedge\tau} \mathbb{E}[P_s|\mathcal{F}_s^Y, \tau = s]\lambda_s\, ds\right)_{t\in[0,T]}$$

is a $(\mathbb{P}, \mathcal{G})$-local martingale. This implies, in view of $(\mathbb{E}[P_\tau \mathbf{1}_{[\tau \le T]}|\mathcal{G}_t])_{t\in[0,T]}$ being a $(\mathbb{P}, \mathcal{G})$-martingale, that the process α in (3.10) is given by

$$\alpha_t = \int_0^{t\wedge\tau} \mathbb{E}[P_s|\mathcal{F}_s^Y, \tau = s]\lambda_s\, ds, \qquad t \in [0, T].$$

The claim now follows directly from (3.10). $\square$

We will next look at some specific examples where the finite variation part in the decomposition of the rebate is of a simpler form.

Example 3.1. In many situations the rebate is $\mathcal{F}^Y$-adapted. In this case,

$$\alpha_t = \int_0^{t\wedge\tau} P_s \lambda_s\, ds.$$

If one is not interested in the Doob–Meyer decomposition but merely the value of the rebate, it is well known (see Proposition 5.1.1 in [3]) that

$$\begin{aligned}\mathbb{E}[P_\tau \mathbf{1}_{[t<\tau\le T]}|\mathcal{G}_t] &= \mathbf{1}_{[\tau>t]}\frac{1}{Z_t}\mathbb{E}\left[-\int_t^T P_u\, dZ_u \middle| \mathcal{F}_t^Y\right] \\ &= \mathbf{1}_{[\tau>t]}\frac{1}{Z_t}\mathbb{E}\left[\int_t^T P_u \lambda_u Z_u\, du \middle| \mathcal{F}_t^Y\right].\end{aligned}$$

Recall from Corollary 2.2 that $Z_t = \exp\left(-\int_0^t \lambda_s\, ds\right)\xi_t^{-1}\kappa_t$ where ξ and κ are as defined in the same corollary. If we further assume the condition of Corollary 2.4, then there exists a probability measure $\tilde{\mathbb{Q}} \sim \mathbb{P}$ such that $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = \xi_T^{-1}\kappa_T$ so that

$$\mathbb{E}[P_\tau \mathbf{1}_{[t<\tau\le T]}|\mathcal{G}_t] = \mathbf{1}_{[\tau>t]}\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_t^T P_u\lambda_u \exp\left(-\int_t^u \lambda_s\, ds\right) du \middle| \mathcal{F}_t^Y\right],$$

which agrees with Proposition 4.3 in [9]. The advantage of the formulas above is that they do not contain the random time τ inside the expectation on the right hand side. However, they are valid only if P is $\mathcal{F}^Y$-adapted.

Example 3.2. Similar to the previous example, if the value of rebate is given by $F(\tau, Y_\tau)$ for some deterministic F, then

$$\alpha_t = \int_0^{t\wedge\tau} F(s, Y_s)\lambda_s ds.$$

The following equation of extrapolation is of interest in its own. Note that the additional assumption that p defined below is continuous is automatically satisfied when $\mathcal{H}$ is a Brownian filtration.

Corollary 3.3. *Let P defined by (3.4) satisfy Assumption 3. Fix a $t \in (0, T]$ and set $p_s := \mathbb{E}[P_t|\mathcal{H}_s]$. Assume further that p is continuous. Then, for any $s \le t$*

$$\mathbb{E}[P_t|\mathcal{G}_s] = \mathbb{E}[P_t] + \int_0^t \left\{\hat{f}_s + \mathbb{E}[P_t b(s, X_s)|\mathcal{G}_s] - \mathbb{E}[P_t|\mathcal{G}_s]\mathbb{E}[b(s, X_s)|\mathcal{G}_s]\right\} d\beta_s$$
$$+ \int_0^t \left\{\hat{p}_{s-} - \mathbb{E}[p_s|\mathcal{F}_s^Y, \tau = s]\right\} dL_s,$$

where f is the $\mathcal{H}$-adapted process satisfying $d[p, B]_t = f_t dt$.

Proof. Note that $\mathbb{E}[P_t|\mathcal{G}_s] = \hat{p}_s$. Since p is a square integrable $(\mathbb{P}, \mathcal{H})$-martingale, it has an orthogonal decomposition of the following form:

$$p_s = \mathbb{E}[P_t|\mathcal{H}_0] + \int_0^s f_r dB_r + \bar{n}_s,$$

where $\bar{n}$ is a square integrable $(\mathbb{P}, \mathcal{H})$-martingale orthogonal to B; see Section 3 of Chapter IV in [32]. The claim now follows from Theorem 3.2. $\square$

3.1. Extensions

Acute reader would have noticed that we had not made use of the Markov property of the vector (X, Y) in the proofs. This makes the extension of the results of this section to a non-Markovian setting an easy task.

Indeed, let τ be an $\mathcal{H}$-stopping time independent of the $\mathcal{H}$-Brownian motion B, and the observation process Y is given by

$$Y_t = B_t + \int_0^t b_s\, ds \tag{3.13}$$

for a progressively measurable process b such that

$$\mathbb{E}\left[\int_0^T b_s^2\, ds\right] < \infty.$$

Suppose that

$$Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t^Y] = 1 + \int_0^t \left\{\mathbb{E}[\mathbf{1}_{[\tau>s]} b_s|\mathcal{F}_s^Y] - Z_s \mathbb{E}[b_s|\mathcal{F}_s^Y]\right\} dB_s^Y - \int_0^t \lambda_s Z_s\, ds$$

for some $\mathcal{F}^Y$-predictable process λ, where

$$B_t^Y = Y_t - \int_0^t \mathbb{E}[b_s|\mathcal{F}_s^Y]\, ds$$

as usual. Then, all the results of this section will continue to hold.

On the other hand, it does not seem easy to relax the assumption that τ and B are independent. The difficulty is not in the computation of the filtering formulas but the existence of an absolutely continuous compensator for Z; see Remark 3.

Appendix

A.1. Proofs of Theorems 2.1 and 2.2 and Lemma 2.1

Proof of Theorem 2.1. Let $\mathbb{Q}_x$ denote the law of the solution of (2.1) with the initial condition $X_0 = x$ and $\mathbb{W}_x$ be the law of the standard Brownian motion starting at x, both being defined on the canonical space $C(\mathbb{R}_+, \mathbb{R})$ where $X_t(\omega) = \omega(t)$ and $\mathcal{F}_t = \sigma(X_s; s \leq t)$.

1. One has, for any $t \geq 0$,

$$\mathbb{Q}_x|_{\mathcal{F}_t} = \exp\left(A(X_t) - A(x) - \frac{1}{2}\int_0^t \left\{a^2(X_s) + a'(X_s)\right\} ds\right) \cdot \mathbb{W}_x|_{\mathcal{F}_t}$$

The fact that $\exp\left(\int_0^t a(X_s)dX_s - \frac{1}{2}\int_0^t a^2(X_s)\, ds\right)$ is a $(\mathbb{W}_x, \mathcal{F})$-martingale follows from the fact that X is the non-exploding solution to (2.1) and from, e.g., Exercise 2.10 in Chapter IX of [33]. Let f be a test function with a support in $[0, T]$ where T is an arbitrary constant. Then,

$$\begin{aligned}
\mathbb{Q}_x[f(\tau)] &= \exp(-A(x))\, \mathbb{W}_x\left[f(\tau) \exp\left(A(X_T) - \frac{1}{2}\int_0^T \left\{a^2(X_s) + a'(X_s)\right\} ds\right)\right] \\
&= \exp(-A(x))\, \mathbb{W}_x\left[\mathbf{1}_{[\tau \leq T]} f(\tau) \exp\left(A(X_T) - \frac{1}{2}\int_0^T \left\{a^2(X_s) + a'(X_s)\right\} ds\right)\right] \\
&= \exp(-A(x))\, \mathbb{W}_x\left[f(\tau) \exp\left(-\frac{1}{2}\int_0^\tau \left\{a^2(X_s) + a'(X_s)\right\} ds\right)\right] \\
&= \exp(-A(x))\, \mathbb{W}_x\left[f(\tau)\mathbb{W}_x\left[\exp\left(-\frac{1}{2}\int_0^\tau \left\{a^2(X_s) + a'(X_s)\right\} ds\right)\Bigg|\tau\right]\right], \qquad \text{(A.1)}
\end{aligned}$$

where the third equality is due to the Optional Sampling Theorem and the fact that f vanishes outside $[0, T]$. Since τ has a density, namely $\ell(\cdot, x)$, under $\mathbb{W}_x$, we conclude from the arbitrariness of T that it has a density under $\mathbb{Q}_x$ as well[2] More precisely,

$$\mathbb{Q}_x[\tau \in dt] = \exp(-A(x))\, \mathbb{E}_x^{(3)} \times \left[\exp\left(-\frac{1}{2}\int_0^t \left\{a^2(X_s) + a'(X_s)\right\} ds\right)\Bigg| X_t = 0\right] \ell(t, x)\, dt,$$

[2] Note that we are not claiming that this density integrates to 1; i.e. τ could be infinite with positive $\mathbb{Q}_x$-probability. An example of this is when $a \equiv 1$, i.e. X is a Brownian motion with a positive drift.

where $\mathbb{E}^{(3)}_x$ is expectation with respect to the law of the 3-dimensional Bessel process starting at x. This is due to the well-known relationship between the law of the Brownian motion conditioned on its first hitting time of 0 and that of 3-dimensional Bessel bridge, which follows from William's time reversal result, see Corollary 4.6 in Chapter VII of [33]. Moreover, $\mathbb{Q}_x[\tau > 0] = 1$ since X is continuous and $x > 0$. This proves $H^a(0, x) = 1$ and the desired absolute continuity of H^a. The strict positivity similarly follows from the fact that $\mathbb{Q}_x \sim \mathbb{W}_x$, when restricted to $\mathcal{F}_t$, and that $\mathbb{W}_x[\tau > t] > 0$ for every $t \geq 0$.

2. In order to prove the second claim note that since $a(x) \geq -K_g(1 + |x|)$, in view of standard comparison results for the solutions of SDEs (see [33]), the solution to (2.1) is always bigger than the solution of

$$dX_t = dW_t - K_g(1 + |X|_t)\, dt.$$

Thus, the solution of (2.1) is larger than the solution to

$$dX_t = dW_t - K_g(1 + X_t)\, dt, \tag{A.2}$$

until the first hitting time of 0 by the latter. Let $\mathbb{Q}^{(-K_g)}_x$ be the law of the solution of (A.2) with the initial condition $X_0 = x$ on the canonical space. Then, by the aforementioned comparison argument we have $\mathbb{Q}_x\left[\frac{1}{\tau} \geq t\right] \leq Q^{(-K_g)}_x\left[\frac{1}{\tau} \geq t\right]$, i.e.

$$\mathbb{E}_x\left[\frac{1}{\tau}\right] \leq \mathbb{E}^{(-K_g)}_x\left[\frac{1}{\tau}\right]. \tag{A.3}$$

Moreover, using the absolute continuity relationship between $Q^{(-K_g)}_x$ and $\mathbb{W}_x$ as above, we obtain

$$\begin{aligned}
&\mathbb{Q}^{(-K_g)}_x[\tau \in dt] \\
&\quad = \exp\left(\frac{K_g}{2}(t + 2x + x^2)\right)\mathbb{E}^{(3)}_x\left[\exp\left(-\frac{K_g^2}{2}\int_0^t (1 + X_s)^2\, ds\right)\middle| X_t = 0\right]\ell(t, x)\, dt \\
&\quad \leq \exp\left(\frac{K_g}{2}(t + 2x + x^2)\right)\mathbb{E}^{(3)}_x\left[\exp\left(-\frac{K_g^2}{2}\int_0^t (1 + X_s^2)\, ds\right)\middle| X_t = 0\right]\ell(t, x)\, dt \\
&\quad \leq \frac{2\exp\left(\frac{K_g}{2}t(1 - K_g)\right)(K_g t)^{3/2}}{\left[\exp(K_g t/2) - \exp(-K_g t/2)\right]^{3/2}} \\
&\qquad \times \exp\left(K_g x - \frac{K_g}{2}x^2\left\{\frac{K_g t\coth(K_g t) - 1}{K_g t} - 1\right\}\right)\ell(t, x)\, dt \\
&\quad = \frac{2\exp\left(-\frac{K_g^2}{2}t\right)(K_g t)^{3/2}}{\left[\exp(K_g t/6) - \exp(-5K_g t/6)\right]^{3/2}} \\
&\qquad \times \exp\left(K_g x - \frac{K_g}{2}x^2\left\{\frac{K_g t\coth(K_g t) - 1}{K_g t} - 1\right\}\right)\ell(t, x)\, dt \\
&\quad \leq 2\delta^{3/2}\exp\left(\frac{K_g}{2}(-K_g t + 2x)\right)\ell(t, x)\, dt,
\end{aligned} \tag{A.4}$$

where the second inequality follows from Formula 2.5 in [34] and the last line is due to the fact that $\frac{y\coth(y)-1}{y} \geq 1$ for $y \geq 0$. Thus,

$$\mathbb{E}_x^{(-K_g)}\left[\frac{1}{\tau}\right] \leq 2\delta^{3/2}\exp\left(K_g x\right)\int_0^\infty \frac{e^{-\frac{K_g^2}{2}t}}{t}\ell(t,x)\,dt.$$

Also note that

$$\int_0^\infty \frac{e^{-\frac{K_g^2}{2}t}}{t}\ell(t,x)\,dt = -\frac{\partial}{\partial x}\int_0^\infty e^{-\frac{K_g^2}{2}t}\frac{1}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}\,dt.$$

As

$$\int_0^\infty e^{-\frac{K_g^2}{2}t}\frac{1}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}}\,dt = \frac{1}{x}e^{-K_g x}.$$

Differentiating above with respect to x in conjunction with (A.3) yields

$$\int_0^\infty \frac{1}{s}\ell^a(s,x)\,ds \leq 2\delta^{3/2}\frac{1+K_g x}{x^2}.$$

3. In order to prove the last assertion, first let

$$\sigma(t,x) := \exp\left(-A(x)\right)\mathbb{E}_x^{(3)}\left[\exp\left(-\frac{1}{2}\int_0^t \left\{a^2(X_s)+a'(X_s)\right\}\,ds\right)\bigg|X_t=0\right]$$

so that $\ell^a(t,x) = \sigma(t,x)\ell(t,x)$. Observe that σ is uniformly bounded, locally in t, if $A(\infty) > -\infty$. Since $t\ell(t,x)$ is uniformly bounded, there is nothing to prove when $A(\infty) > -\infty$.

When $A(\infty) = -\infty$, we must have $a(\infty) < \infty$. Then, there are two cases to consider: either $a(\infty) > -\infty$ and, consequently, a is bounded on $[0,\infty]$, or $a(\infty) = -\infty$. We will prove the claim in the latter case. The case of bounded a is easier and can be handled by the change of measure technique that we will employ below.

Suppose $a(\infty) = -\infty$ and let $\mathbb{U}_x^k$ be the law of the Ornstein–Uhlenbeck process, which is the unique solution to

$$X_0 = x + B_t - k\int_0^t X_s\,ds.$$

Then, by an application of Girsanov theorem, one has

$$\begin{aligned}
&\mathbb{Q}_x[\tau\in dt]\\
&\quad= \mathbb{U}_x^{K_a}\left[\exp\left(\int_0^t \{a(X_s)+K_aX_s\}\,dX_s\right.\right.\\
&\qquad\left.\left.+\frac{1}{2}\int_0^t\left\{K_a^2X_s^2 - a^2(X_s)\right\}ds\right)\bigg|X_t=0\right]\mathbb{U}_x^{K_a}(\tau\in dt)\\
&\quad= \mathbb{U}_x^{K_a}\left[\exp\left(-F(x)+\frac{1}{2}\int_0^t\left\{K_a^2X_s^2 - a^2(X_s)\right.\right.\right.\\
&\qquad\left.\left.\left.-a'(X_s)-K_a\right\}ds\right)\bigg|X_t=0\right]\mathbb{U}_x^{K_a}(\tau\in dt)\\
&\quad\leq K\exp\left(-F(x)\right)\mathbb{U}_x^{K_a}(\tau\in dt),
\end{aligned} \tag{A.5}$$

for some constant K, depending on t, in view of Assumption 1, where

$$F(x) := \int_0^x \{a(y) + K_a y\} dy.$$

Observe that under Assumption 1, for large values of x, $\exp(-F(x)) \leq \exp(cx^p)$ for some constant c, and $p < 2$. On the other hand,

$$t\mathbb{U}_x^{K_a}[\tau \in dt] = \frac{x}{\sqrt{2\pi}} \left(\frac{K_a t}{\sinh(K_a t)}\right)^{\frac{3}{2}} \exp\left(\frac{K_a}{2}\left(t - x^2(\coth(K_a t) - 1)\right)\right);$$

see, e.g. [18]. Since $\frac{x}{\sinh x}$ is bounded and $\coth(K_a t) > 1$ when $t \leq N$, for any N, claim follows. $\square$

Proof of Theorem 2.2. The idea of the proof is to apply the nonlinear filtering formulas to find an expression for Z which will lead to the statement of the theorem after Fubini type arguments as explained below. This will be done in three steps.

Step 1. We will first prove that

$$Z_t = \mathbb{E}[H^a(t, X_0)] + \int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]} H^a(t-s, X_s)\left(b(s, X_s) - \mathbb{E}[b(s, X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right] dB_s^Y, \tag{A.6}$$

and Z is strictly positive. To this end, let $P_s := \mathbf{1}_{[\tau>s]} H^a(t-s, X_s)$ for $s \leq t$. It follows from (2.2) and the Markov property of X that, for any t, $(P_s)_{s\in[0,t]}$ is a bounded, continuous and nonnegative $(\mathbb{P}, \mathcal{H})$-martingale with $P_\tau = 0$ on the set $[\tau \leq t]$ and $P_t = \mathbf{1}_{[\tau>t]}$. Since

$$\int_0^t \mathbb{E}^2[b(s, X_s)]\, ds \leq K_b^2(t) \int_0^t \mathbb{E}X_s^2\, ds < \infty \tag{A.7}$$

in view of Remark 1, it follows from Theorem 8.1 in [30] that for $s \leq t$

$$\bar{P}_s = \mathbb{E}[H^a(t, X_0)] + \int_0^t \left\{\mathbb{E}[P_r b(r, X_r)|\mathcal{F}_r^Y] - \bar{P}_r \mathbb{E}[b(r, X_r)|\mathcal{F}_r^Y]\right\} dB_r^Y,$$

where $\bar{P}$ is the $\mathcal{F}^Y$-optional projection of P, and the innovation process defined by

$$dB_t^Y = Y_t - \mathbb{E}[b(t, X_t)|\mathcal{F}_t^Y],$$

is an $\mathcal{F}^Y$-Brownian motion. Noticing that $Z_t = \bar{P}_t$ yields the claimed representation.

In order to show the strict positivity we will make use of the process M defined in (2.8). Observe from the discussion following (2.8) that M^{-1} is a strictly positive $(\mathbb{P}, \mathcal{H})$-martingale, and $\mathbb{Q}_t \sim \mathbb{P}_t$ is a probability measure on $\mathcal{H}_t$ defined by

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}_t} = M_t^{-1},$$

under which $(Y_s)_{s\in[0,t]}$ is a standard Brownian motion independent of $(X_s)_{s\in[0,t]}$. Also observe that the laws of $(X_s)_{s\in[0,t]}$ under $\mathbb{P}_t$ and $\mathbb{Q}_t$ are the same since the measure change only affects Y. Moreover, in view of (2.11), one has

$$Z_t \mathbb{E}^{\mathbb{Q}_t}[M_t|\mathcal{F}_t^Y] = \mathbb{E}^{\mathbb{Q}_t}[\mathbf{1}_{[\tau>t]} M_t|\mathcal{F}_t^Y].$$

Since M is strictly positive, so is $\mathbb{E}^{\mathbb{Q}_t}[M_t|\mathcal{F}_t^Y]$; thus, strict positivity of Z is equivalent to that of $\mathbb{E}^{\mathbb{Q}_t}[\mathbf{1}_{[\tau>t]} M_t|\mathcal{F}_t^Y]$. However, for any $A \in \mathcal{F}_t^Y$ with $\mathbb{Q}_t[A] > 0$, $\mathbb{Q}_t[A, \tau > t] = \mathbb{Q}_t[A]\mathbb{Q}_t[\tau > t] > 0$ since $\mathbf{1}_{[\tau>t]}$ is independent of $\mathcal{F}_t^Y$ under $\mathbb{Q}_t$, and $\mathbb{Q}_t[\tau > t] = \mathbb{P}[\tau > t] > 0$ in view of

Part 1 of Theorem 2.1. Thus, $\mathbb{E}^{\mathbb{Q}_t}[\mathbf{1}_{[\tau>t]}M_t|\mathcal{F}_t^Y] > 0$, $\mathbb{Q}_t$-a.s. since M_t is strictly positive $\mathbb{Q}_t$-a.s. Claim now follows from the equivalence of $\mathbb{Q}_t$ and $\mathbb{P}_t$.

Step 2. Next, we will show that

$$\begin{aligned}&\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}H^a(t-s,X_s)\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\quad=\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\qquad-\int_0^t\left(\int_0^s \mathbb{E}\Big[\mathbf{1}_{[\tau>r]}\ell^a(s-r,X_r)\right.\\ &\qquad\left.\times\left(b(r,X_r)-\mathbb{E}[b(r,X_r)|\mathcal{F}_r^Y]\right)\big|\mathcal{F}_r^Y\Big]dB_r^Y\right)ds. \qquad (A.8)\end{aligned}$$

Recall that

$$H^a(t-s,X_s)=1-\int_s^t \ell^a(u-s,X_s)\,du;$$

thus,

$$\begin{aligned}&\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}H^a(t-s,X_s)\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\quad=\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\qquad-\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}\int_s^t \ell^a(u-s,X_s)\,du\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\quad=\int_0^t\left\{\mathbb{E}\left[\mathbf{1}_{[\tau>s]}b(s,X_s)|\mathcal{F}_s^Y\right]-Z_s\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right\}dB_s^Y\\ &\qquad-\int_0^t\int_s^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}\ell^a(u-s,X_s)\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]du\,dB_s^Y,\end{aligned}$$

where the interchange of expectation and integration is justified by Fubini's theorem since ℓ^a is positive and integrable, b is Lipschitz, and $\mathbb{E}|X_s|<\infty$ for any $s\geq 0$.

Moreover, if we can interchange the order of stochastic and ordinary integrals in the second integral above, we can further write

$$\begin{aligned}&\int_0^t \mathbb{E}\left[\mathbf{1}_{[\tau>s]}H^a(t-s,X_s)\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\right]dB_s^Y\\ &\quad=\int_0^t \eta_s\,dB_s^Y-\int_0^t\left(\int_0^u \mathbb{E}\Big[\mathbf{1}_{[\tau>s]}\ell^a(u-s,X_s)\right.\\ &\qquad\left.\times\left(b(s,X_s)-\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\right)\big|\mathcal{F}_s^Y\Big]dB_s^Y\right)du.\end{aligned}$$

This interchange of ordinary and stochastic integrals can be justified by Theorem 65 in Chapter IV of [32] if

$$\mathbb{E}\left[\int_0^t\int_0^s \mathbf{1}_{[\tau>r]}(\ell^a(s-r,X_r))^2X_r^2\,dr\,ds\right]<\infty \qquad (A.9)$$

since b is locally Lipschitz and $b(t,0)=0$ by Assumption 2. Since all the terms are positive we have

$$\begin{aligned}\int_0^t\int_0^s \mathbf{1}_{[\tau>r]}(\ell^a(s-r,X_r))^2X_r^2\,dr\,ds &= \int_0^t\int_r^t \mathbf{1}_{[\tau>r]}(\ell^a(s-r,X_r))^2X_r^2\,ds\,dr\\ &\leq K\int_0^t \mathbf{1}_{[\tau>r]}X_r^2\int_r^t\frac{1}{s-r}\ell^a(s-r,X_r)\,ds\,dr\\ &\leq K\int_0^t \mathbf{1}_{[\tau>r]}X_r^2\int_r^\infty\frac{1}{s-r}\ell^a(s-r,X_r)\,ds\,dr\\ &\leq K\int_0^t \mathbf{1}_{[\tau>r]}(1+K_gX_r)dr\\ &\leq K\int_0^t (1+K_g|X_r|)dr\end{aligned}$$

where the second line is due to $\ell(u,x)<K\frac{1}{u}$ by Theorem 2.1 for some constant K, possibly depending on t, and the fourth line is a consequence of (2.3). (A.9) now follows from Remark 1.

Step 3. Combining (A.6) and (A.8) yields

$$\begin{aligned}Z_t &= \mathbb{E}[H^a(t,X_0)]+\int_0^t\eta_s\,dB_s^Y\\ &\quad-\int_0^t\left(\int_0^s\mathbb{E}\Big[\mathbf{1}_{[\tau>r]}\ell^a(s-r,X_r)\Big(b(r,X_r)-\mathbb{E}[b(r,X_r)|\mathcal{F}_r^Y]\Big)\Big|\mathcal{F}_r^Y\Big]dB_r^Y\right)ds.\end{aligned}$$

The proof is now complete since

$$\mathbb{E}[H^a(t,X_0)]=1-\int_0^t\int_0^\infty\ell^a(u,x)\mu(dx)\,du. \quad\square$$

Proof of Lemma 2.1. Observe that for any bounded $\mathcal{F}_t^Y$-measurable random variable F, we can write, in view of the absolute continuity relationship between $\mathbb{P}$ and $\mathbb{Q}$,

$$\mathbb{E}[F]=\mathbb{E}^{\mathbb{Q}}[M_tF]=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]F\right].$$

Since $(\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y])_{t\in[0,T]}$ is a strictly positive $(\mathbb{Q},\mathcal{F}^Y)$-martingale, the above implies that

$$d\mathbb{P}|_{\mathcal{F}_t^Y}=\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]d\mathbb{Q}|_{\mathcal{F}_t^Y},$$

and $\mathbb{P}|_{\mathcal{F}_t^Y}\sim\mathbb{Q}|_{\mathcal{F}_t^Y}$. Moreover, since Y is $\mathbb{Q}$-Brownian motion, we have from the predictable representation property of Brownian filtrations that

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}_t^Y]=1+\int_0^t\phi_s\mathbb{E}^{\mathbb{Q}}[M_s|\mathcal{F}_s^Y]\,dY_s,$$

for some $\mathcal{F}^Y$-predictable process ϕ since $(\mathbb{E}^{\mathbb{Q}}[M_s|\mathcal{F}_s^Y])_{s\in[0,T]}$ is strictly positive and continuous; hence predictable. Next note that $\mathcal{F}^Y$-canonical decomposition of Y under $\mathbb{P}$ is given by

$$Y_t=B_t^Y+\int_0^t\mathbb{E}[b(s,X_s)|\mathcal{F}_s^Y]\,ds$$

by an application of Theorem 8.1 in [30]. Girsanov Theorem now tells us that $\phi_s = \mathbb{E}[b(s, X_s)|\mathcal{F}^Y_s]$, i.e.

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}^Y_t] = 1 + \int_0^t \mathbb{E}^{\mathbb{Q}}[M_s|\mathcal{F}^Y_s]\mathbb{E}[b(s, X_s)|\mathcal{F}^Y_s]\, dY_s.$$

Moreover, since $\mathbb{E}[b(s, X_s)|\mathcal{F}^Y_s] = \frac{\mathbb{E}^{\mathbb{Q}}[M_s b(s,X_s)|\mathcal{F}^Y_s]}{\mathbb{E}^{\mathbb{Q}}[M_s|\mathcal{F}^Y_s]}$, we also have

$$\mathbb{E}^{\mathbb{Q}}[M_t|\mathcal{F}^Y_t] = 1 + \int_0^t \mathbb{E}^{\mathbb{Q}}[M_s b(s, X_s)|\mathcal{F}^Y_s]\, dY_s.$$

In view of (2.10), an application of Ito's formula yields that

$$\mathbb{E}[M_t^{-1}|\mathcal{F}^Y_t] = 1 - \int_0^t \mathbb{E}[M_s^{-1}|\mathcal{F}^Y_s]\mathbb{E}[b(s, X_s)|\mathcal{F}^Y_s]\, dB^Y_s. \quad \square$$

A.2. A measure theoretic lemma

In this section, we will state and prove a lemma which will be useful in obtaining the main filtering result of this paper contained in Theorem 3.2. The proof is based on elementary measure theoretic methods. We refer the reader to Section 5 in Chapter IV of [33] for equivalent definitions of optional projections, which will be used in the proof below. In Lemma below $\mathbb{Q}$ is the probability measure on $\mathcal{H}_{\tau\vee T}$ which is equivalent[3] to the restriction of $\mathbb{P}$ to $\mathcal{H}_{\tau\vee T}$ and under which $Y^{\tau\vee T}$ is a stopped Brownian motion independent of $X^{\tau\vee T}$. In what follows, $\mathcal{B}$ denotes the class of Borel sets and we suppress the dependency on T to ease notation when no confusion arises.

Lemma A.1. 1. *Let $T > 0$ be a fixed real number and suppose that F is a $\mathcal{B}([0, T]) \otimes (\sigma(\tau) \vee \mathcal{H}_T)$-measurable and $\mathbb{Q}$-integrable stochastic process. Denote the $(\mathcal{F}^Y_t)_{t\in[0,T]}$-optional σ-algebra with $\mathcal{O}^Y$ and let $\mathcal{F}^{Y,\tau}$ be the smallest filtration satisfying the usual conditions and including $(\mathcal{F}^Y_t)_{t\in[0,T]}$ such that $\sigma(\tau) \subset \mathcal{F}^{Y,\tau}_0$. Then, there exists a function $f : [0, \infty) \times (\Omega \times [0, T)) \mapsto \mathbb{R}$ such that f is $\mathcal{B}([0, \infty)) \otimes \mathcal{O}^Y$-measurable and the $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-optional projection of F is given by $(f(\tau(\omega), \omega, t))_{t\in[0,T]}$.*
2. *For every $u \geq 0$ and $t \geq 0$, $\mathbb{E}^{\mathbb{Q}}[F_t|\mathcal{F}^Y_t, \tau = u] := f(u, \cdot, t)$ is $\mathcal{F}^Y_t$-measurable, where f is as above.*

Proof. 1. Note that we can assume without any loss of generality that F is $\mathcal{B}([0, T)) \otimes (\sigma(\tau) \vee \mathcal{F}^X_T \vee \mathcal{F}^Y_T)$-measurable in view of the tower property of conditional expectations. Let $\mathcal{I}$ and $\mathcal{C}$ denote the class of $\mathbb{Q}$-integrable and $\mathcal{B}([0, T)) \otimes (\sigma(\tau) \vee \mathcal{F}^X_T \vee \mathcal{F}^Y_T)$-measurable stochastic processes, and the class of $\mathcal{B}([0, \infty)) \otimes \mathcal{O}^Y$-measurable real-valued functions defined on $[0, \infty) \times (\Omega \times [0, T))$, respectively. For $F \in \mathcal{I}$, let us denote its $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-optional projection with ${}^oF$ and define

$$\mathcal{R} := \left\{F \in \mathcal{I} : {}^oF = (f(\tau(\omega), \omega, t))_{t\in[0,T]},\ f \in \mathcal{C}\right\}.$$

$\mathcal{R}$ is clearly a vector space containing constant functions. Moreover, if $(F^n)_{n\geq 1} \subset \mathcal{R}$ is a sequence of uniformly bounded and increasing processes such that $\lim_{n\to\infty} F^n = F$, then

[3] As before $\mathbb{Q}$ is defined via the Radon–Nikodym derivative $M_{\tau\vee T}$. Observe that M is still a martingale until the finite stopping time $\tau \vee T$ in view of the same no-explosion argument used in the beginning of the proof of Theorem 2.1. Also recall from the discussion following the definition of M in (2.8) the impossibility of defining an equivalent $\mathbb{Q}$ on $\mathcal{H}_\infty$.

$F \in \mathcal{R}$, as well. Indeed, let f^n denote the measurable function corresponding to ${}^oF^n$ for each n. Then, $f := \liminf_{n\to\infty} f^n$ belongs to $\mathcal{C}$ since $f^n \in \mathcal{C}$ for each n. Moreover, for any $\mathcal{F}^{Y,\tau}$-stopping time S, which is necessarily less than or equal to T,

$$\begin{aligned} f(\tau(\omega), \omega, S(\omega)) &= \liminf_{n\to\infty} f^n(\tau(\omega), \omega, S(\omega)) \\ &= \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}}\left[F^n_S|\mathcal{F}^{Y,\tau}_S\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[F_S|\mathcal{F}^{Y,\tau}_S\right], \end{aligned}$$

where the second equality follows from the definition of optional projections and the last equality follows from the Dominated Convergence Theorem. This shows that ${}^oF = (f(\tau(\omega), \omega, t))_{t\in[0,T]}$ and, thus, $F \in \mathcal{R}$. Consequently, $\mathcal{R}$ is a monotone vector space.

In order to prove the claim using a monotone class argument, it suffices to prove the statement for a multiplicative class generating $\mathcal{I}$. Such a class is provided by the processes

$$F_t(\omega) = \mathbf{1}_{[0,s)}(t)F^1(\omega)F^2(\omega)F^3(\tau(\omega)),\ 0 \le s \le T,\ F^1 \in L^\infty(\mathcal{F}^X_T),\ F^2 \in L^\infty(\mathcal{F}^Y_T)$$

and F^3 is a bounded Borel measurable function on $[0, \infty)$. Let $(f^2(\omega, t))_{t\in[0,T]}$ be the càdlàg version of the $(\mathbb{Q}, \mathcal{F}^Y)$-martingale $(\mathbb{E}^{\mathbb{Q}}[F^2|\mathcal{F}^Y_t])_{t\in[0,T]}$ and note that f^2 is an $\mathcal{O}^Y$-measurable function. Moreover, $(f^2(\omega, t))_{t\in[0,T]}$ is also a $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-martingale since $(Y_t)_{t\in[0,T]}$ and τ are independent under $\mathbb{Q}$. Therefore, the $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-optional projection of F^2 is given by f^2 in view of the Optional Stopping Theorem.

Also observe that there exists a Borel measurable function, f^1, such that $f^1(\tau) = \mathbb{E}^{\mathbb{Q}}[F^1|\tau]$. We will now see that ${}^oF = (\mathbf{1}_{[0,s)}(t)f^1(\tau(\omega))f^2(\omega, t)F^3(\tau(\omega)))_{t\in[0,T]}$. Clearly, $(\mathbf{1}_{[0,s)}(t)f^1(\tau(\omega))f^2(\omega, t)F^3(\tau(\omega)))_{t\in[0,T]}$ is an $\mathcal{F}^{Y,\tau}$-optional process since f^2 is $\mathcal{F}^Y$-optional. In order to show it is the desired optional projection, it suffices to show that for any $\mathcal{F}^{Y,\tau}$-stopping time S

$$\mathbb{E}^{\mathbb{Q}}[F_S] = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)f^1(\tau)f^2(\omega, S)F^3(\tau(\omega))\right].$$

Indeed,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)F^1F^2F^3(\tau)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)\mathbb{E}^{\mathbb{Q}}\left[F^1|\mathcal{F}^Y_T, \tau\right]F^2F^3(\tau)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)\mathbb{E}^{\mathbb{Q}}\left[F^1|\tau\right]F^2F^3(\tau)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)f^1(\tau)F^2F^3(\tau)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{[0,s)}(S)f^1(\tau)f^2(\omega, S)F^3(\tau)\right], \end{aligned}$$

where the second equality is due to the independence of $X^{\tau\vee T}$ and Y^T under $\mathbb{Q}$ and the last equality holds since f^2 is the $(\mathbb{Q}, \mathcal{F}^{Y,\tau})$-optional projection of F^2.

Finally, since we have already observed that f^2 is an $\mathcal{O}^Y$-measurable function, it now easily follows that the function $f(u, \omega, t) := \mathbf{1}_{[0,s)}(t)f^1(u)f^2(\omega, t)F^3(u)$ belongs to $\mathcal{C}$. The Monotone Class Theorem now yields that any bounded member of $\mathcal{I}$ is contained in $\mathcal{R}$. The general case follows from applying the Dominated Convergence Theorem to F and the sequence $((F \wedge n) \vee -n)$.

2. Note that the u-section, $f(u, \cdot, \cdot)$, of f is $\mathcal{O}^Y$-measurable for each $u \ge 0$ since f is measurable with respect to the product σ-algebra. In particular, $(f(u, \omega, t))_{t\in[0,T]}$ is $\mathcal{F}^Y$-adapted for each $u \ge 0$. $\square$

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